MATH 551 - Problem Set 9

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October 30, 2014

1. Consider $x \in S^2$ such that T_x, that is, the tangent space of x, shares the same normal vector as P (that is, in a sense, x is a pole of A). This will be our candidate for point $C \in S^2$ which is equidistant from all points on A. Now consider the great circle, G, of S^2 which is parallel to (does not intersect with) the "latitude" A (note that in the case where A is a great circle of S^2, we choose A to be G and our argument will still hold). Now, because A and G are parallel, we know that x must too be a pole of G. Because x is a pole of G and G is a great circle, we know that the distance from x to any point on G is $\frac{\pi}{2}$, and because A and G are parallel we know that the spherical segments connecting points on A and G are of equal length, call that length c (side note: this can be seen by choosing any great circle, L, on S^2, and connecting the points of A and G only by using the spherical segments whose poles are on L). Now, we know that the distance between x and G as $\frac{\pi}{2} = r + c$, where c is the constant mentioned earlier, well, solving for r we have $r = \frac{\pi}{2} - c$ and because we know that c is constant, we can conclude that r is constant (and because the max value of c is $\frac{\pi}{2}$, we know $r > 0$). Thus, we have shown that x is exactly the point C we were looking for, and therefore A is a circle in spherical geometry. $\square$

2. Let the components of v be v_1, v_2, v_3. Because we know that vectors in $v^{\perp}$ are all only those vectors $x =< x_1, x_2, x_3 >$ that satisfy $v_1x_1 + v_2x_2 - v_3x_3 = 0$ where v_1, v_2, v_3 are constants, we see that a choice of any two x components immediately fixes the third (e.g. $x_3 = \frac{v_1x_1+v_2x_2}{v_3}$), that is, there are only two degrees of freedom, which means that the set of x's span a plane (alternatively, one may notice that, because v_1, v_2, v_3 are constants, the equation becomes the equation for a plane in standard form).

3. We will employ that: $\sinh(x) + \cosh(x) = e^x$, $\sinh(-x) = -\sinh(x)$, and $\cosh(x) = \cosh(-x)$. We start with $\sinh(x+y) = \frac{e^{x+y} - e^{-x-y}}{2} = \frac{e^x e^y - e^{-x} e^{-y}}{2}$, but we notice that we can rewrite each of these e^{var} terms using the identity stated earlier, so we achieve

$$\frac{(\sinh(x)+\cosh(x))(\sinh(y)+\cosh(y)) - (\sinh(-x)+\cosh(-x))(\sinh(-y)+\cosh(-y))}{2}$$

which, by two of the identities above ($\sinh(-x) = -\sinh(x)$ and $\cosh(x) = \cosh(-x)$) is equal to

$$\frac{(\sinh(x)+\cosh(x))(\sinh(y)+\cosh(y)) - (-\sinh(x)+\cosh(x))(-\sinh(y)+\cosh(y))}{2}$$

now multiplying out we have $\frac{1}{2}(\sinh(x)\sinh(y) + \sinh(x)\cosh(y) + \cosh(x)\sinh(y) + \cosh(x)\cosh(y) - \sinh(x)\sinh(y) + \sinh(x)\cosh(y) + \cosh(x)\sinh(y) - \cosh(x)\cosh(y))$. We notice that the 1st and 5th term cancel as well as the 4th and 8th, and the remaining 4 can be combined into two so we have $\frac{2\sinh(x)\cosh(y) + 2\cosh(x)\sinh(y)}{2}$, and the 2's cancel so we may conclude that $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$.

4. Suppose $v \neq 0$ and $\Phi(v, u) = 0, \forall u \in R^3$. Well, that would mean that $\exists v_1, v_2, v_3$, not all zero such that $\forall u_1, u_2, u_3 \in R,\ v_1 u_1 + v_2 u_2 - v_3 u_3 = 0$. Well, suppose you have such u_1, u_2, u_3, that means that $u_3 = \frac{v_1 u_1 + v_2 u_2}{v3}$, but because u_3 can vary freely, independent of u_1, u_2, then u_3 may be chosen so that this equation does not hold (for instance, simply add 1 to u_3 and the equation will no longer hold). So therefore one of our assumptions was wrong, namely that $v \neq 0$, and thus we must have that $v = 0$.

5. Because we know that there exists a bijection between H^2 and K (where $K = (x_1, x_2, 1) \in R^3 \mid x_1^2 + x_2^2 < 1)$, a proof of Desargues in K will suffice to prove Desargues in H^2. We continue with our proof for Desargues in Euclidean geometry from class, as the interior of K is a Euclidean space with the only modified condition that now all points be $\in K$.

$P = k_1 A + (1-k_1)A'$, $P = k_2 B + (1-k_2)B'$, and $P = k_3 C + (1-k_3)C'$ (by theorem 1). Thus (by taking the difference of the last two) $0 = k_2 B + (1-k_2)B' - k_2 C + (1-k_3)C' \Rightarrow \frac{k_3 C - k_2 B}{k_3 - k_2} = \frac{(1-k_2)B' - (1-k_3)C'}{k_3 - k_2} \Rightarrow L = \frac{k_3}{k_3 - k_2} C - \frac{k_2}{k_3 - k_2} B. \Rightarrow (k_3 - k_2)L = k_3 C - k_2 B$. By similarity, we see that the same is the case for M and K, that is we have the following:

$$(k_3 - k_2)L = k_3 C - k_2 B$$

$$(k_1 - k_3)M = k_1 A - k_3 C$$

$$(k_2 - k_1)K = k_2 B - k_1 A$$

Now the sum of these equations means that $k_2 - k_2)L + (k_1 - k_3)M + (k_2 - k_1)K = 0$ and we notice that the sum of the coefficients $(k_3 - k_2) + (k_1 - k_3) + (k_2 - k_1) = 0$, and therefore we may conclude by theorem 2 that K, L, and M are collinear.